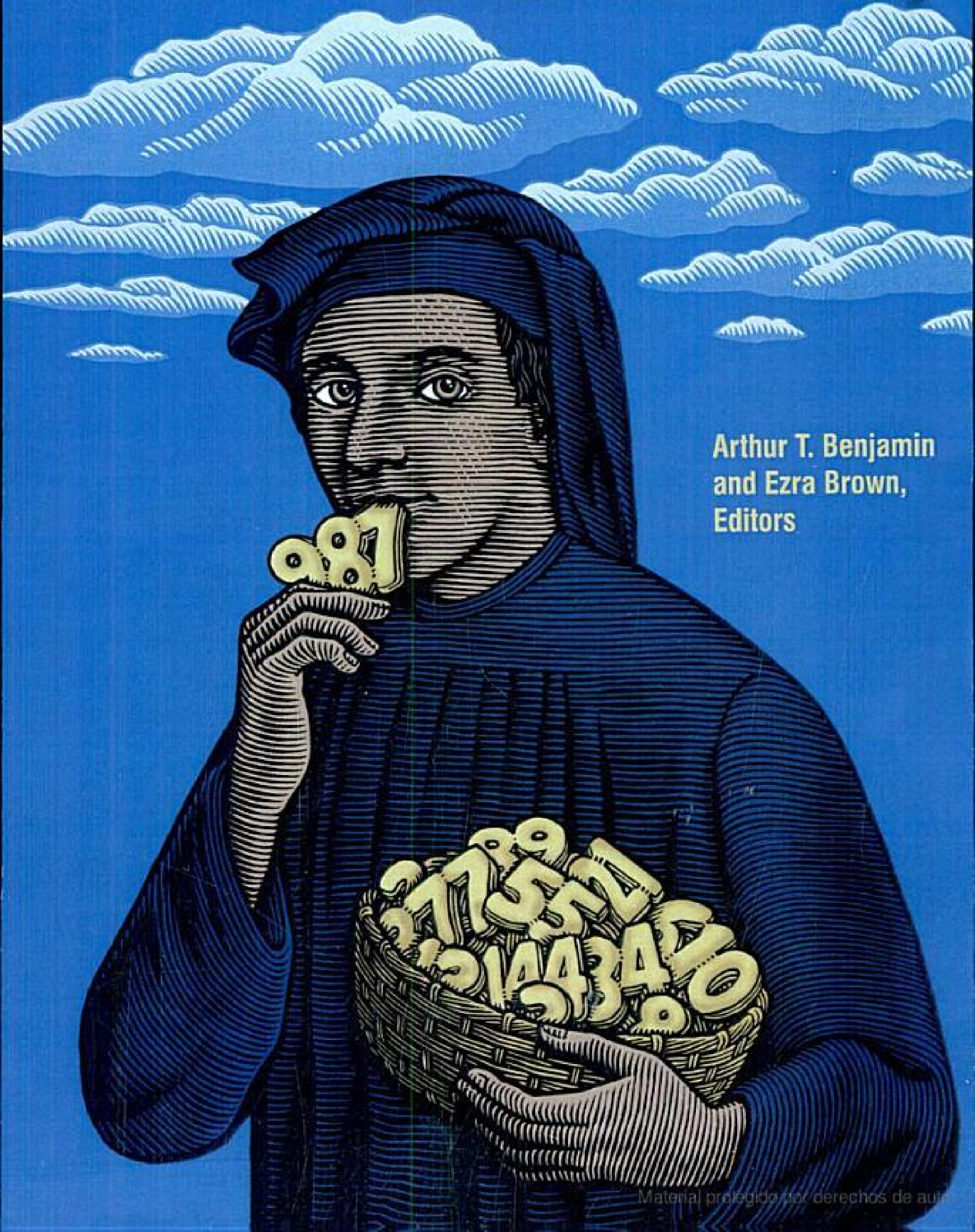
Mathematical Association of America

DOLCIANI MATHEMATICAL EXPOSITIONS #34

Biscuits of Number Theory



Arthur T. Benjamin
and Ezra Brown,
Editors

Biscuits of Number Theory

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Biscuits of Number Theory

Edited by
Arthur T. Benjamin
and
Ezra Brown

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Teoria de numeros
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Numeros de Fibonacci

The DOLCIANI MATHEMATICAL EXPOSITIONS series of the Mathematical Association of America was established through a generous gift to the Association from Mary P. Dolciani, Professor of Mathematics at Hunter College of the City University of New York. In making the gift, Professor Dolciani, herself an exceptionally talented and successful expositor of mathematics, had the purpose of furthering the ideal of excellence in mathematical exposition.

The Association, for its part, was delighted to accept the gracious gesture initiating the revolving fund for this series from one who has served the Association with distinction, both as a member of the Committee on Publications and as a member of the Board of Governors. It was with genuine pleasure that the Board chose to name the series in her honor.

The books in the series are selected for their lucid expository style and stimulating mathematical content. Typically, they contain an ample supply of exercises, many with accompanying solutions. They are intended to be sufficiently elementary for the undergraduate and even the mathematically inclined high-school student to understand and enjoy, but also to be interesting and sometimes challenging to the more advanced mathematician.

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29. *Conics*, Keith Kendig
30. *Hesiod's Anvil: falling and spinning through heaven and earth*, Andrew J. Simoson
31. *A Garden of Integrals*, Frank E. Burk
32. *A Guide to Complex Variables* (MAA Guides #1), Steven G. Krantz
33. *Sink or Float? Thought Problems in Math and Physics*, Keith Kendig
34. *Biscuits of Number Theory*, edited by Arthur T. Benjamin and Ezra Brown

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Introduction

> "Powdermilk Biscuits: Heavens, they're tasty and expeditious! They're made from whole wheat, to give shy persons the strength to get up and do what needs to be done." — Garrison Keillor, *A Prairie Home Companion*

You are probably wondering, "What exactly are biscuits of number theory?" In this book, we have an assortment of articles and notes on number theory, where each item is not too big, easily digested, and makes you feel all warm and fuzzy when you're through. We hope they will whet your appetite for more! Overall, we felt that the biscuit analogy hit the spot (in addition, one of the editors bakes biscuits for his students).

In this collection, we have chosen articles that we felt were exceptionally well-written and that could be appreciated by anyone who has taken (or is taking) a first course in number theory. This book could be used as a textbook supplement for a number theory course, especially one that requires students to write papers or do outside reading. Here are some of the possibilities:

- Each piece in the collection is a fine starting point for classroom discussions. After telling the Ramanujan story about sums of two cubes, an instructor can ask the class what questions this story might prompt. Students can be given a list of cubes and asked to verify Ramanujan's claim. Some of the material in Silverman's Taxicabs article can show them how following up on an innocent little remark can lead to some quite wonderful mathematics.
- The articles can be used as follow-ups to classroom presentations. For example, after introducing the Fibonacci numbers, have some students read the Tanton paper and assign one or more of the "Taking it Further" questions. Have others read the two matched "Fibonacci numbers—exposed" papers and describe how they are the same and how they are different.
- Students who see the standard proofs of the infinitude of the primes or the Quadratic Reciprocity Law and who are curious about other proofs can be directed to the alternative proofs in this collection and encouraged to present them to the class. This could lead to students' searching for other proofs in the literature and writing them up in a paper.
- Students in most courses in number theory learn about Paul Erdős. Have them read Erdős' paper in this collection, work out the details, and present them to the class, along with a determination (with proof) of the instructor's Erdős number.
- Zagier's one-sentence proof of the Two-Squares Theorem gives students an opportunity to work out the details of a very short proof. In doing this, they learn the mathematics behind that proof. Another possibility is for students to present Jackson's one-line proof that every prime $p = 8n + 3$ is of the form $a^2 + 2b^2$, and perhaps create proofs of their own for similar results.
- Many of the papers contain material that might give students a start on an undergraduate research project.
- For students who wonder how to do research in number theory, have them read

Pólya's translation of Euler's paper on sums of divisors, so they can see how one of the masters did it.

- The Arithmetic section contains articles that can give future secondary-school teachers ideas on how to introduce their students to patterns hidden in powers of two and fractions and to pictures hidden in numbers.

We have just mentioned many of the biscuits by title, but all 40 of them have interesting mathematics for number theorists, young and not-so-young. After some of the articles, we provide "second helpings" where readers can learn about further developments and other topics related to the article.

This project began at the MAA Northeastern Section meeting in November, 2004. During a break between talks, the two editors chatted about a possible invited paper session for MathFest 2005 in Albuquerque. Ten minutes later, we had a topic and a format: four speakers, each with a half-hour slot, would present what we called "Gems of Number Theory."

A half-hour after that, we had a list of potential speakers who would exactly fill the bill. A few weeks later, we had our program. The large and appreciative audience at MathFest 2005 heard Marc Chamberland speak on the Collatz $3x + 1$ Problem, Ed Burger on Diophantine Approximation, Jennifer Beineke on Great Moments of the Riemann zeta Function, and Roger Nelsen on Visual Gems of Number Theory. The two of us decided that the session was a success.

The MAA thought so, too. Not long after MathFest 2005, we got a request from Don Van Osdol of the MAA encouraging us to put together a book based on the four session talks. The four speakers at our session thought this was a fine idea and agreed to contribute their work.

We then set about looking for other articles that would be suitable for an undergraduate audience. We began with a list of all of the recent MAA prize-winning papers on number theory. Then, we sent emails to current and recent editors of the MAA journals, some well-known number theorists, and a selection of recent MAA presidents, asking them for suggestions; the response was both overwhelming and gratifying. We read through many back issues of the MAA journals and the AMS Notices. At one point, the number of suggestions was up into seven bits!

Then the whittling-down process began, and it was not easy. For one thing, no sooner had we brought the list down to a manageable size than one of us would toss a few more into the mix—and we'd start over again. Finally, the list converged to the collection you have before you.

And what a collection it is. Prizes won by these papers include the Allendoerfer, Chauvenet, Conant, Trevor Evans, Lester Ford, Hasse and Pólya Awards. Among the authors are recipients of the MAA's Haimo and Alder Awards for Distinguished Teaching, as well as Martin Gardner, George Pólya, Paul Erdős, and Leonhard Euler. We have included many visual delights, and some of the proofs are very likely proofs from "The Book", Erdős' imaginary compendium of perfect proofs.

We have divided the collection into seven chapters: Arithmetic, Primes, Irrationality, Sums of Squares and Polygonal Numbers, Fibonacci Numbers, Number Theoretic Functions, and Elliptic Curves, Cubes, and Fermat's Last Theorem. Before each chapter, we provide the reader with a summary of the articles that appear there. (You might call this an "aroma.") As with any anthology, you don't have to read the Biscuits in order. Dip into them anywhere: pick something from the Contents that strikes your fancy, and have at it. If the end of an article leaves you wondering what happens next, then by all means dive in and do some research. You just might discover something new!

We would like to acknowledge the following people for their advice and assistance with this book: Don Albers, Jerry Alexanderson, Tom Apostol, Jennifer Beineke, Lowell Beineke, Ed Burger, Marc Chamberland, Karl Dilcher, Underwood Dudley, Frank Farris, Ron Graham, Richard Guy, Deanna Haunsperger, Roger Horn, Christopher Hughes, Dan Kalman, Steve Kennedy, Hugh Montgomery, Roger Nelsen, Ken Ono, Bruce Palka, Carl Pomerance, Peter Ross, Martha Siegel, Jim Tattersall, Don Van Osdol, Stan Wagon, and Paul Zorn.

Finally, we wish to thank our families for their support. It has been a pleasure working with MAA putting this collection together. We are particularly grateful to Elaine Pedreira, Beverly Ruedi, and especially Rebecca Elmo for their hard work, high standards, and dedication to this project.

— Arthur Benjamin

— Ezra Brown

August 8, 2008

Part I: Arithmetic

In Chapter XI of Lewis Carroll's *Alice's Adventures in Wonderland*, the Mock Turtle, in describing his schooling to Alice, referred to the four branches of arithmetic as Ambition, Distraction, Uglification and Derision. Our first six Biscuits will treat all of these and more.

We begin with Jim Tanton's "A dozen questions about the powers of two" (*Math Horizons*, vol. 8, no. 1 (September 2001), pp. 5-10), an Evans Award-winning exploration of the surprising amount of interesting mathematics to be found among the powers of two. In Tanton's inimitable style, a dozen questions about the powers of two lead to answers–and even more questions. He asks questions about weighing problems, sums of truncated triangular numbers, Egyptian multiplication, checkers in a circle, a variant of the game Survivor, folding fractions, leading digits, and more. Most answers include a challenge to "Take It Further." Read this most enjoyable paper with pencil and paper in hand, and then explain the following complete sequence: 4, 1, 5, 2, 9, 6, 46, 3, 53.

Early in number theory classes, we learn that 30 is the largest integer n such that every positive integer strictly between 1 and n and relatively prime to n is a prime. In 1907 H. Bonse gave a clever proof that hinges on the fact that the product of the first n prime numbers exceeds the square of the $(n + 1)^{st}$ prime. Very neat, but what comes next? Is there a largest number n such that every positive integer less than n and relatively prime to n is a prime power? Do some exploring and guess. Then read our next Biscuit, in which Michael Dalezman strengthens Bonse's Inequality and answers this and other similar questions in "From 30 to 60 is not twice as hard" (*Math. Magazine*, vol. 73, no. 2 (April 2000), pp. 151–153). At the end of this paper, Dalezman invites you to verify that 2730 is the largest integer n such that all positive integers less than n and prime to n have at most two prime divisors, multiplicities included. Replace "two" by "three" and "four", and the answers are 210210 and 29099070, respectively.

We add the fractions $\frac{a}{c}$ and $\frac{b}{d}$ by finding the least common denominator, adjusting the numerators, and adding. If $\frac{a}{c}$ and $\frac{b}{d}$ are reduced fractions then, depending on the values of c and d, the resulting fraction might never, sometimes, or always be further reducible. The answer is "never" for $\frac{a}{10}$ and $\frac{b}{12}$, "sometimes" for $\frac{a}{15}$ and $\frac{b}{21}$, and "always" for $\frac{a}{14}$ and $\frac{b}{38}$. Harris Shultz and Ray Shiflett were at a workshop where this came up, and they wondered what the pattern was. In their article "Reducing the sum of two fractions" (*Mathematics Teacher*, vol. 98, no. 7 (March 2005), pp. 486–490), they tell us what they found and gently lead us to the answers. (Again, make a guess, then read the article.) At the end, they encourage their readers to explore deeper waters: "It would be interesting to know how much of the facts about reducing numerical fractions carry over to the class of rational functions." It would, indeed!

Let n be a positive integer, and let f be a mapping on $\mathbb{Z}$ mod n, the integers mod n.

For $x \in \mathbb{Z}$ mod n, the forward orbit of x under f is the set $\{x_1 = x, x_2 = f(x_1), x_3 = f(x_2) \ldots\}$. Now, join the points (x_1, x_1), (x_1, x_2), (x_2, x_2), (x_2, x_3), ... The picture you get should look familiar, because this graphical iteration method generates pictures similar to those that appear in every book on dynamical systems. Rafe Jones and Jan Pearce use this method to tackle a modest goal in our next Biscuit—nothing less than bringing the beauty of fractions to the postmodern era of America's visually-oriented, quantitatively illiterate culture. They begin their Allendoerfer Award-winning paper, "A postmodern view of fractions and the reciprocals of Fermat primes" (*Math. Magazine*, vol.73, no. 2, (April 2000), pp. 83–97), by reminding us that the amazing fractal images of chaos theory took popular culture by storm. Wouldn't it be nice, they wonder, if this transformation of chaotic dynamical systems from differential equations, analysis and algebra into posters and tee shirts of the Mandelbrot set can be applied to, say, fractions?

They do not simply wonder. They apply to the study of certain fractions the graphical techniques that were used to transform the Mandelbrot set from algebra to cultural icon. The resulting paper is a picture-filled treatment of such topics as rotational graph pairs, rotational symmetry, fractions with prime denominators, and perfectly symmetric numbers. At the end, they apologize that they were not able to include all of their striking images in the paper—but they provide a pointer to their web site.

Mathematics, including number theory, is the science of patterns, and nothing helps us better in discovering new patterns than a good picture. In "Visible structures in number theory" (*Amer. Math. Monthly*, vol. 108, no. 10 (December 2001), pp. 897–910), Peter Borwein and Loki Jörgenson present pleasing periodic patterns provided by polynomials, Pascal's triangle, and π. This paper, which was awarded the MAA's Lester Ford Award, presents a number of open questions that are provoked by these pictures, including the meta-mathematical question "What does it mean to prove a theorem visually or experimentally?"

If you read MAA journals, you know that Roger Nelsen is the preeminent collector of Proofs Without Words (PWWs) anywhere. Our final arithmetical Biscuit is a selection of his "Visual gems of number theory" (*Math Horizons*, vol. 15, no. 3 (February 2008), pp. 7–9, 31). These visual treats include characterizations of primitive Pythagorean triples, Euclid's perfect number formula, relationships between triangular numbers and squares, a proof that $\sqrt{2}$ is irrational, and Larson's elegant wordless proof—surely one for Paul Erdős' mythical "Book" of perfect proofs —that every triangular number is a "choose two" binomial coefficient. Read them and marvel, then try your hand at making up some yourself. For example, the sum of the first m even squares $2^2 + 4^2 + \cdots + (2m)^2$ is a "choose three" binomial coefficient. Find a PWW of this.

A Dozen Questions About the Powers of Two

James Tanton

Everyone is familiar with the powers of two: 1, 2, 4, 8, 16, 32, 64, 128, and so on. They appear with surprising frequency throughout mathematics and computer science. For example, the number of subsets of a finite set is a power of two, as too is the sum of the entries of any row of Pascal's triangle. (Mathematically, these two statements are the same!) The largest prime number known today is one less than a power of two, a cube of tofu can be sliced into a maximum of 2^n pieces with n planar cuts, and every even perfect number is the sum of consecutive integers from 1 up to one less than a power of two!

Here I have put together a dozen curiosities all about the powers of two. These puzzles toy with results and ideas from classic number theory and geometry, game theory, and even popular TV culture (one problem is about a variation of the game *Survivor*)! I hope you enjoy thinking about them as much as I did.

1. A Weighty Problem

A woman possesses five stones, each weighing an integral number of kilograms. She claims, with the use of a simple see-saw balance, she can match the weight of any stone you give her and thereby determine its weight. She makes this claim under the proviso that your stone is of integral weight and weighs no more than 31 kilograms.

What are the weights of her five stones?

2. Multiplication without Multiplying

Here's an alternative method to long multiplication: Head two columns with the numbers you wish to multiply. Progressively halve the figures in the left-hand column (ignoring remainders) while doubling the numbers on the right. Continue this operation until the left-hand column is reduced to 1. Delete all rows with an even number in the left-hand column and add all the surviving numbers from the right-hand column. This sum is the desired product.

73	×	23
~~36~~		~~46~~
~~18~~		~~92~~
9		184
~~4~~		~~368~~
~~2~~		~~736~~
1		1472
		1679

$$73 \times 23 = 1679.$$

Why does this work?

3. Truncated Triangular Numbers

The numbers 5, 12, and 51, for example, can be written as a sum of two or more consecutive positive integers:

$$5 = 2 + 3$$
$$12 = 3 + 4 + 5$$
$$51 = 6 + 7 + 8 + 9 + 10 + 11.$$

Which numbers *cannot* be written as a sum of at least two consecutive positive integers?

4. Survivor

N people, numbered from 1 to N, are stranded on an island. They play the following variation of the TV game *Survivor*:

Members of the group vote whether person number N should survive or be escorted off the island. If 50% or more of the people agree to this person's survival then the game ends here and the N people all take an equal share of a $1,000,000 cash prize. If, on the other hand, the Nth person is voted off the island, the remaining $N - 1$ people will take a second vote to determine the survival of the $(N - 1)$th player (again with a quota of 50%). They do this down the line until a vote eventually passes and a person survives. The cash prize is then shared equally among all the folks remaining after this successful vote.

Assume that all players are greedy, but rational, thinkers; that they will always vote for their own survival, for example, and will vote for the demise of another player provided it does not lead to their own demise as a consequence.

The question here is simple: who survives?

5. Pascal Curiosity

Prove that all entries in the 2^nth row ($n \in \mathbf{N}$) of Pascal's triangle are odd.

6. Checkers in a Circle

Betty places a number of black and white checkers in arbitrary order on the circumference of a circle. (Say Betty lays down N checkers.) Charlie now places new checkers between the pairs of adjacent checkers in Betty's ring: he places a white checker between every two that are the same color, a black checker between every pair of opposite color. He then removes Betty's original checkers to leave a new ring of N checkers in a circle.

Betty then performs the same operation on Charlie's ring of checkers, following the same rules. The two players alternate performing this maneuver over and over again on the circle of checkers before them. Show that if N is a power of two, all the checkers will eventually be white, no matter the arrangement of colors Betty initially puts down.

7. Classic Number Theory

Is $2^{91} - 1$ prime? What about $2^{91} + 1$?

8. De Polignac's Remarkable Conjecture

In the mid-nineteenth century, the French mathematician A. de Polignac made a remarkable observation: It seems that every odd number larger than one can be written as a sum of a power of two and a prime.